

# 1 Standard Deviation Calculation

## 1.1 Notation

$$\mathbb{M}_{\mathbb{B}}(s) \equiv \begin{pmatrix} \mathbb{B}_{-0}^m(s)' & 0 & \cdots & 0 \\ 0 & \mathbb{B}^m(s)' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{B}^m(s)' \end{pmatrix}_{(p+1) \times (m-1+mp)},$$

and

$$Q_{(k),zz} = \frac{1}{N_k T} \sum_{i \in G_{k|K}} \sum_{t=1}^T \ddot{z}_{it} \ddot{z}'_{it},$$

$$\mathbb{S}_{(k)}(s) = \frac{\sigma_{v(k)}^2}{m} \mathbb{M}_{\mathbb{B}}(s) Q_{(k),zz}^{-1} \mathbb{M}_{\mathbb{B}}(s)'$$

In addition, write

$$\tilde{f}_{i(k)}(\varrho) \equiv \tilde{f}(y_i | x_i; c_1, \delta_{u1}^2, c_2, \delta_{u2}, \tau, \vartheta_{(k|K^*)}), \text{ and}$$

$$\hat{f}_{i(k)}(\varrho) \equiv \tilde{f}(y_i | x_i; c_1, \delta_{u1}^2, c_2, \delta_{u2}, \tau, \hat{\vartheta}_{(k|K^*)}),$$

for short, and

$$\mathbb{I} \equiv -E \left[ \sum_{k=1}^{K^*} \frac{N_k}{N} \cdot \frac{\partial^2}{\partial \varrho \partial \varrho'} \log \tilde{f}_{i(k)}(\varrho) \right] \Bigg|_{\varrho=\varrho^0}.$$

## 1.2 SE for $\hat{\sigma}_v^2$

I ignore  $k$  in the following. An estimate of  $\text{Var}(v_{it}^2)$  can be:

$$\widehat{\text{Var}}(v_{it}^2) = \frac{1}{N(T-1)} \sum_{i=1}^N \sum_{t=1}^T (\hat{v}_{it}^2 - \overline{\hat{v}_{it}^2})^2,$$

where

$$\overline{\hat{v}_{it}^2} = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \hat{v}_{it}^2.$$

Then the standard error for  $\hat{\sigma}^2$  can be calculated as

$$\frac{\sqrt{\widehat{\text{Var}}(v_{it}^2)}}{\sqrt{NT}}.$$

### 1.3 SE for $\hat{\varrho}$

Recall that

$$\mathbb{I} \equiv -E \left[ \sum_{k=1}^{K^*} \frac{N_k}{N} \cdot \frac{\partial^2}{\partial \varrho \partial \varrho'} \log \tilde{f}_{i(k)}(\varrho) \right] \Bigg|_{\varrho=\varrho^0}.$$

If we denote the log-likelihood function as

$$\mathcal{L}(\varrho) = \sum_{i=1}^N \log \tilde{f}_{i(k)}(\varrho),$$

then

$$\hat{\mathbb{I}} = - \frac{\partial^2 \mathcal{L}(\varrho)}{\partial \varrho^2} \Bigg|_{\varrho=\hat{\varrho}}.$$

Note that  $\mathcal{L}(\varrho)$  is the objective function for you to find  $\hat{\varrho}$ .

We can obtain an estimate of  $\hat{\mathbb{I}}$  by numerical derivatives. Suppose  $\varrho$  is  $p \times 1$ . Denote a  $p \times 1$  vector

$$e_l = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow l\text{-th element}$$

and  $l = 1, \dots, p$ . Set a small value  $h = 10^{-5}$  (you can choose another small value that generates a reasonable result). Note we only care about diagonals for inferences. The first diagonal can be estimated as

$$-\frac{\widehat{\partial^2 \mathcal{L}(\varrho)}}{\partial \varrho_1^2} \Bigg|_{\varrho=\hat{\varrho}} = -\frac{\mathcal{L}(\hat{\varrho} + he_1) - 2\mathcal{L}(\hat{\varrho}) + \mathcal{L}(\hat{\varrho} - he_1)}{h^2}$$

due to

$$\begin{aligned} f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x+h/2) - f'(x-h/2)}{h} \approx \frac{f'(x+h/2) - f'(x-h/2)}{h} \\ &\approx \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \end{aligned}$$

For other diagonals, it can be obtained similarly

$$\frac{\widehat{\partial^2 \mathcal{L}(\varrho)}}{\partial \varrho_l^2} \Bigg|_{\varrho=\hat{\varrho}} = -\frac{\mathcal{L}(\hat{\varrho} + he_l) - 2\mathcal{L}(\hat{\varrho}) + \mathcal{L}(\hat{\varrho} - he_l)}{h^2}, l = 1, \dots, p.$$